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Analytical solution for the potential flow through the wall of *n*-sided hollow cylinders of regular polygonal cross-section

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Abstract

The calculation of the potential flow through the wall of hollow cylinders of regular polygonal cross-section with the inner and the outer boundaries considered to be lines of different but constant potential is a classical problem in the field of non-viscous transport phenomena. The known empirical correlation's result from numerical and experimental investigations and are thus restricted to limited ranges of variables, while a more universal analytic solution has not yet been published, as far as we know. In this paper we do present such an analytical solution together with asymptotes and a simple approximation, that does not need higher mathematical functions, for easy application.

1. Introduction

Consider *n*-sided hollow cylinders of regular polygonal cross-section shown in Fig. 1 (n = 3,..., 6 for example) with their inner and outer boundaries at constant but different dimensionless potentials ϕ. Within the walls the Laplace-equation

$$\frac{\partial^2\phi}{\partial\,\mathrm{Re}(z)^2}+\frac{\partial^2\phi}{\partial\,\mathrm{Im}(z)^2}=0 \tag{1}$$

shall apply. The quantity to be calculated here is the amount of potential flow per unit length

$$\Psi=\int\left(\frac{\partial\phi}{\partial\mathbf{m}}\right)\mathrm{d}\xi \tag{2}$$

perpendicular to the isopotential of the isosceles trapezoidal plane domain (see Fig. 2) which is by reason of symmetry of the basic shape. The answer has direct applications to calculations of transport phenomena like heat-flow, inviscid fluid flow and electrical flow.

There are three ways to treat the problem: the experimental, the numerical and the analytical way. Experimental investigations have been done by Langmuir [1] in 1913 and by Smith et al. [2] in 1958 from the view of heat transfer for hollow cylinders of quadratic cross-section (n = 4, $\alpha = 1/4$). Smith also treated the problem numerically by a relaxation method, but only made calculations for $x=(1/y)=2$

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Nomenclature

A	non-dimensional length of an isopotential line
C_1, C_2	complex numbers
F	hypergeometric function
$\mathrm{Im}(c)$	imaginary part of the complex number c
k	modulus in elliptic integral and hypergeometric function
K	elliptic integral of the first kind
m	co-ordinate perpendicular to an isopotential boundary
n	number of polygon-sides
p	geometric parameter in the w-plane
q	parameter for the generalised geometric mean [Eq. (27)]
R^*	resistance of inverse flow $R^*=y/\Psi$
$\mathrm{Re}(c)$	real part of the complex number c
S	non-dimensional length of a streamline
t	transformed variable
w	point in the w-plane, complex number
y	geometric parameter in the z-plane, ratio of the inner to the outer surface
z	complex co-ordinate.

Greek symbols

α	unit half-circle fraction of the acute angle of the trapezoid
ϕ	dimensionless potential
ψ	dimensionless local flow per unit length
Ψ	dimensionless integral flow per unit length.

Subscripts

a	approximation
c	corner section
r	rectangular section
0	asymptotic ($y \to 0$).

with two grids of different density. Based on these results Hahne and Grigull [4] recommended the empirical correlation

$$\Psi(\alpha = 1/4) = \begin{cases} \dfrac{2\pi}{0.93\ \ln(x) - 0.0502} & (x = (1/y) > 1.4) \\ \dfrac{2\pi}{0.785\ \ln(x)} & (x = (1/y) < 1.4) \end{cases} \tag{3}$$

which can also be found in the 'Wärmeatlas' [3] and is regarded to be the 'state of the art'-solution to this problem in the field of heat-transfer. From our own numerical calculations with finite elements and finite differences and simple analytical deliberations in [5], we proposed

$$\Psi(\alpha = 1/4) = \frac{8}{\left(1 + \dfrac{1-y}{4}\right) \ln\left(\dfrac{1}{y}\right)}, \tag{4}$$

which has the advantage of being continuous. Although (4) fits our numerical calculations with a relative error better than 0.005 for the computed range of $x = 1/y$ from 1.01 to 10, it diverges from (3) for y towards unity. Aware of the fact that experiments and numerical methods each have their disadvantages and encouraged by the remark of a reviewer on [5] we tried to find an analytical solution with the help of confor-

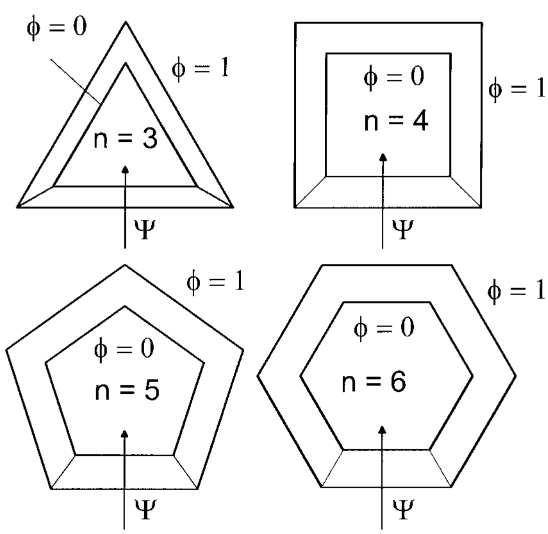

Fig. 1. Sketch of the problem.

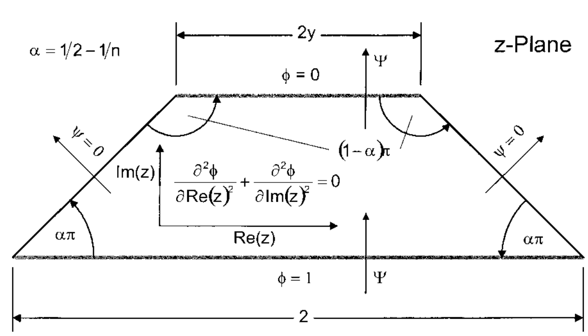

Fig. 2. Basic domain for calculation.

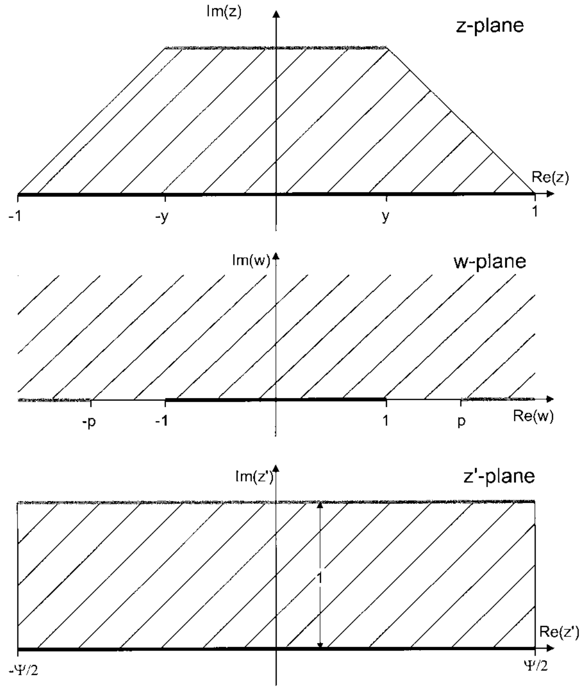

Fig. 3. Mapping of the original problem from the z-plane via the w-plane onto a rectangle in the z'-plane.

mal mapping using Schwarz–Christoffel integrals. In this paper we present this new solution, which is much more general than the earlier numerical one (see [5]) by the introduction of the angle $\alpha\pi$ (see Fig. 2).

2. Analytical solution

The problem described above can be solved if we are able to map the original problem conformally onto a rectangle for which the solution of (1) is known. This is possible by the use of a Schwarz–Christoffel integral that maps the trapezoid conformally from the z-plane to the upper half of the w-plane, in a way that each corner of the trapezoid in the z-plane refers to a point on the real axis of the w-plane. We then use another Schwarz–Christoffel integral to map the upper half of the w-plane back onto a rectangle in the z'-plane (see Fig. 3).

The implicit mapping-formulae are

$$z = C_1 \int_0^w \frac{\mathrm{d}w}{(w^2-1)^{1-\alpha}(w^2-p^2)^{\alpha}} + C_2, \tag{5}$$

for mapping the trapezoid onto the upper half of the w-plane and

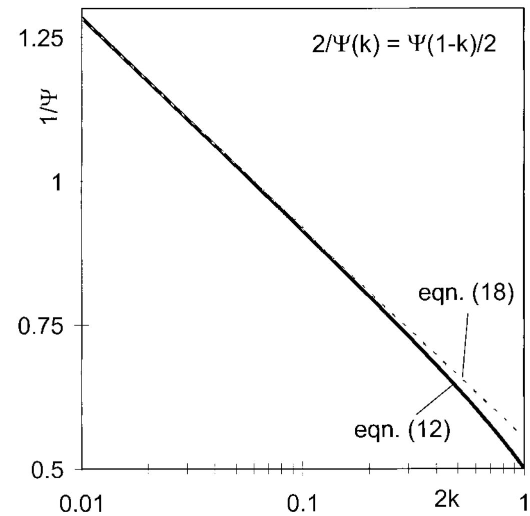

Fig. 4. Flow Ψ vs modulus k.

$$z' = C'_1 \int_0^w \frac{\mathrm{d}w}{(w^2-1)^{1/2}(w^2-p^2)^{1/2}} + C'_2, \tag{6}$$

for mapping the upper half of the w-plane onto the rectangle. The constants C_1, C'_1, C_2, C'_2 are complex numbers to scale and move the domains within the planes. The length of the shorter (inner) baseline $2y$ (ratio of the inner to the outer surface) of the trapezoid is

$$2y = 2C_1 \int_p^{\infty} \frac{\mathrm{d}w}{(w^2-1)^{1-\alpha}(w^2-p^2)^{\alpha}} \tag{7}$$

and for the outer baseline we get

$$2 = 2C_1 \int_0^1 \frac{\mathrm{d}w}{(w^2-1)^{1-\alpha}(w^2-p^2)^{\alpha}}. \tag{8}$$

This yields the following expression for the ratio of the inner to the outer surface of the tube y

$$y = \frac{\displaystyle\int_p^{\infty} \frac{\mathrm{d}w}{(w^2-1)^{1-\alpha}(w^2-p^2)^{\alpha}}}{\displaystyle\int_0^1 \frac{\mathrm{d}w}{(w^2-1)^{1-\alpha}(w^2-p^2)^{\alpha}}} = \frac{J_1}{J_2} \tag{9}$$

and it is clear, that neither C_1 nor C_2 are of any importance here. As the solution for the flow Ψ through the rectangle simply is the length of the isopotential boundaries divided by the length of the streamline boundaries, a similar procedure gives

$$\Psi = 2\frac{\displaystyle\int_0^1 \frac{\mathrm{d}w}{(w^2-1)^{1/2}(w^2-p^2)^{1/2}}}{\displaystyle\int_1^p \frac{\mathrm{d}w}{(w^2-1)^{1/2}(w^2-p^2)^{1/2}}}. \tag{10}$$

Again the constants cancel. Eqs. (9) and (10) can be considered as an analytical solution for the flow in integral representation.

With the substitution of the parameter p by

$$p = 1/k^2 \tag{11}$$

and the help of Maple V Release 5 Eq. (10) can be solved to give

$$\Psi = 2\frac{K(\sqrt{k})}{K(\sqrt{1-k})} \tag{12}$$

wherein K is the complete elliptic integral of the first kind (see [6], p. 590) and k runs from zero to one. Fig. 4 shows the graph of Eq. (12) together with an asymptotic function that will be derived later in the text.

To simplify Eq. (9) we rewrite the integral J_2 in the denominator with the substitution

$$t = w^2 \tag{13}$$

and with Eq. (11) we get

$$\begin{aligned} J_2 &= \frac{k^\alpha}{2}\int_0^1 \frac{\mathrm{d}t}{t^{1/2}(1-t)^{1-\alpha}(1-kt)^\alpha} \\ &= \frac{k^\alpha}{2}\,\frac{\Gamma(1/2)\Gamma(\alpha)}{\Gamma(1/2+\alpha)}F\left(\alpha,\ \frac{1}{2},\ \frac{1}{2}+\alpha,\ k\right) \end{aligned} \tag{14}$$

wherein F is the hypergeometric function (see [6], p. 558) and Γ is the complete gamma function defined by Gauss (see [7], p. 92).

To simplify J_1 in the numerator of Eq. (9) we introduce the substitution

$$u = \frac{w^2 - p^2}{w^2 - 1} \tag{15}$$

and with Eq. (11) we get the solution

$$\begin{aligned} J_1 &= \frac{k^{1/2}}{2}\int_0^1 \frac{\mathrm{d}u}{u^\alpha(1-u)^{1/2}(1-ku)^{1/2}} \\ &= \frac{\sqrt{k}}{2}\frac{\Gamma(1-\alpha)\Gamma(1/2)}{\Gamma(3/2-\alpha)}F\left(\frac{1}{2},\ 1-\alpha,\ \frac{3}{2}-\alpha,\ k\right). \end{aligned} \tag{16}$$

From Eqs. (14) and (16) the geometric ratio y is found to be

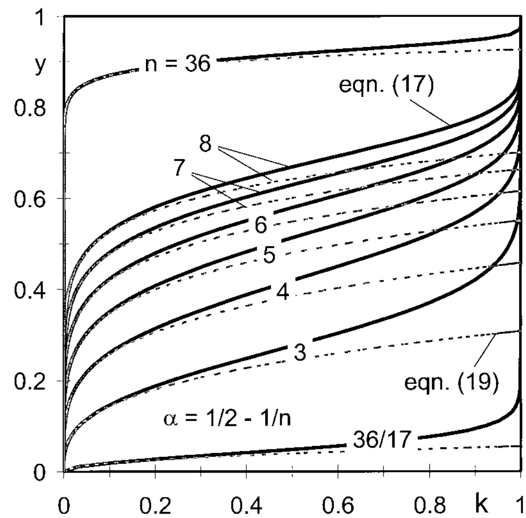

Fig. 5. Geometric parameter y vs modulus k.

$$y = \frac{\cot(\pi\alpha)}{1/2-\alpha}\,\frac{\Gamma^2(1/2+\alpha)}{\Gamma^2(\alpha)}k^{(1/2)-\alpha} \times \frac{F\left(\frac{1}{2},\ 1-\alpha,\ \frac{3}{2}-\alpha,\ k\right)}{F\left(\alpha,\ \frac{1}{2},\ \frac{1}{2}+\alpha,\ k\right)}. \tag{17}$$

In Fig. 5 the graph of Eq. (17) is shown together with an asymptotic function derived in Eq. (19). The Eqs. (12) and (17) represent a parametric solution in k. Usually one is only interested in the flow Ψ depending on the geometric parameters y and α. Therefore Fig. 6 shows a graph of the resistance $R^* = y/\Psi$ vs y for different angles. It was calculated with the help of

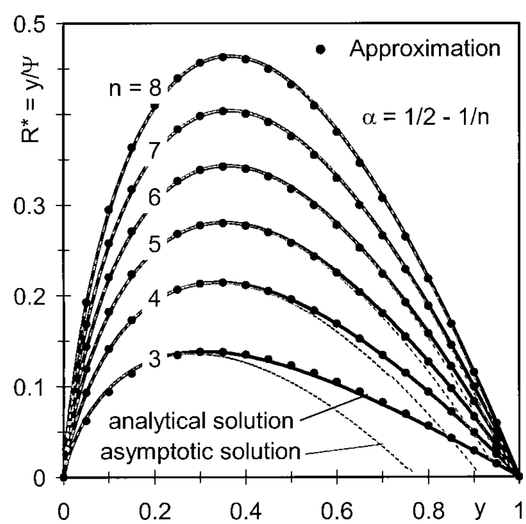

Fig. 6. Resistance R^* vs geometric parameter y.

Maple V Release 5. The choice of plotting $R^*(y)$ in place of $\Psi(y)$ has the advantage, that this quantity remains finite in the whole range $0 \le y \le 1$. For the same reason we preferred to use the geometric ratio y in place of $x = 1/y$, which had been used in the earlier paper [5].

3. Discussion and conclusion

3.1. Asymptotic functions

For k against zero, y tends to zero. This limit can be considered as a line-source in the center of a regular polygonal cylinder. In this case a series evaluation of (12) and (17) for k towards zero yields

$$\lim_{k\to 0}(\Psi(k)) = \frac{2\pi}{4\ \ln(2) - \ln(k)} \tag{18}$$

and

$$\lim_{k\to 0}(y(k)) = \frac{\Gamma(1/2+\alpha)^2 k^{1/2-\alpha}}{\tan(\pi\alpha)(1/2-\alpha)\Gamma^2(\alpha)} \tag{19}$$

if only the first term of the series is considered. From these equations

$$\Psi_0 = \frac{(1-2\alpha)\pi}{\ln(1/y) - \ln\left(\tan(\pi\alpha)\dfrac{1-2\alpha}{2}\,\dfrac{\Gamma^2(\alpha)}{\Gamma^2(1/2+\alpha)}\right) + 2(1-2\alpha)\ln(2)} \tag{20}$$

follows, as an explicit, asymptotic function for the flow through the trapezoidal domain when y tends to zero. Graphs of the Eqs. (18)–(20) are shown in the Figs. 4–6.

The asymptotic function Eq. (20) fulfils the limiting case for α towards zero (i.e. n to infinity), when the trapezoid becomes an infinitesimal part of the circular tube and the flow is given by

$$\lim_{n\to\infty}(n\Psi_0) = \frac{2\pi}{\ln(1/y)}. \tag{21}$$

For k against unity we were not able to derive an asymptotic function for $\Psi(y)$ as the hyper-geometric function has a singularity in this limit.

3.2. Approximation

While the analytical solution is exact and easy to evaluate with the help of sophisticated mathematical programs for a single value, one may not always have such tools at hand. To compute a whole series of values (as necessary in optimisation procedures) it may still take too long on a personal computer. The reason for this becomes clear from Fig. 7. When the geometric parameter y approaches unity, the connected parameter k in the w-plane is even much closer to unity and $\epsilon = -\log(1-k)$ in Fig. 7 is, for $\epsilon > 10$, in good approximation the minimum number of digits needed for successful computation with Maple V Release 5. On the other hand Eq. (20) only holds for small values of y in the general case and still needs the Gamma function. Therefore we tried to derive an approximating

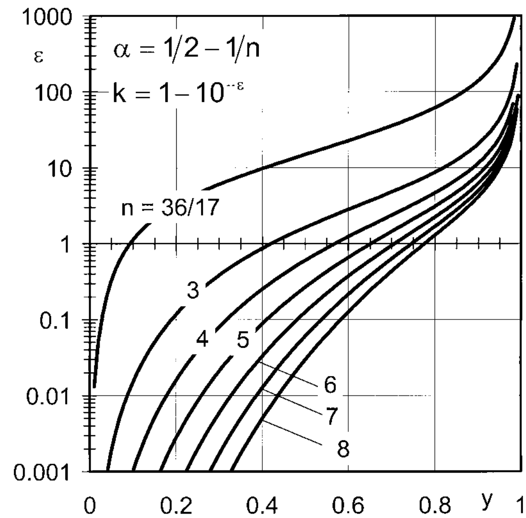

Fig. 7. Digits needed for successful computation.

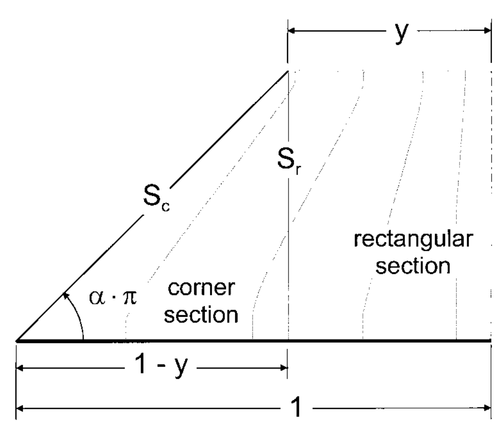

Fig. 8. Geometric considerations for the approximation.

function that allows us to perform quick calculations with a pocket-calculator or a spread-sheet program.

As in [5] we consider the flow to be

$$\Psi = \frac{\bar{A}}{\bar{S}}, \tag{22}$$

wherein $\bar{A}$ is the average length of an isopotential line (or surface per unit length) and $\bar{S}$ is the length of the average flow path. For $\bar{A}$ we assume the logarithmic mean between the length of the inner and the length of the outer isopotential boundaries

$$\bar{A} = \frac{2(1-y)}{\ln(1/y)} \tag{23}$$

as an approximation, which is exact in the limit as n tends to infinity, while for the average length $\bar{S}$ of the flowpath we choose a weighted mean value (see Fig. 8)

$$\bar{S} = yS_r + (1-y)\bar{S}_c \tag{24}$$

between the height of the rectangular section S_r

$$S_r = (1-y)\tan(\pi\alpha) \tag{25}$$

and a generalised geometric mean for the flow path in the corner section $\bar{S}_c$

$$\bar{S}_c = S_c^{(1-q)} S_r^q = \frac{1}{(\sin(\pi\alpha))^q} S_r. \tag{26}$$

In Eq. (26) the exponent q is found to be

$$q = f(\alpha) \tag{27}$$

a function of the acute angle of the trapezoid and by comparison with the exact values from Eqs. (12) and (17) we get

$$q = \frac{1}{1 - c\ \ln(2\alpha)} \tag{28}$$

wherein the constant c was fitted by minimisation of the squares of the errors to be

$$c = 0.75. \tag{29}$$

This finally yields

$$\Psi_a = \frac{2\ \cot(\pi\alpha)}{\ln(1/y)} \frac{1}{y + (1-y)/(\sin(\pi\alpha))^q}, \quad q = \frac{1}{1 - 0.75\ \ln(2\alpha)} \tag{30}$$

as a good approximation (see Fig. 6, black dots), which fulfils the limiting cases including the fact, that for α towards zero (i.e. n to infinity) the trapezoid becomes an infinitesimal part of the circular tube and the flow is given by

$$\lim_{n\to\infty}(n\Psi) = \lim_{n\to\infty}(n\Psi_a) = \frac{2\ \lim(n\ \cot(\pi\alpha))}{\ln(1/y)} = \frac{2\pi}{\ln(1/y)}, \quad \alpha = \frac{1}{2} - \frac{1}{n}. \tag{31}$$

The geometric parameter y, in this limit, is the inner diameter of the tube divided by its outer diameter. For $y > 0.25$ Eq. (26) is in any case within about $\pm 1\%$ from the exact value by Eqs. (12) and (17) and for $n > 3$ it is even much better. For $y < 0.25$ the asymptotic solution should be used for simple calculation.

3.3. Conclusions

With the analytical solution derived above it is possible to calculate the potential flow through several shapes.

The basic geometry in Fig. 2 can be extended by connecting the identical geometry to a boundary in a way that the resulting interior boundary remains a streamline or a line of constant potential. For n-sided polygonal tube as in Fig. 1 for example, we get:

$$\Psi_n = n\Psi(y, \alpha), \quad \alpha = 1/2 - 1/n. \tag{32}$$

The solution even holds if the boundary lines of constant potential and the streamline boundaries are swapped, if the reciprocal value of the flow is taken.

3.4. Note

In case of heat transfer the boundary condition of constant wall temperature considered here is fulfilled for condensing and evaporating processes. In other cases, like a boundary condition of the third kind, application of the solution may still be reasonable if the wall temperature is only a function of the axial coordinate. This is usually assumed in the case of a circular tube with fluids flowing through and around the tube. But this assumption may not be valid for polygonal tubes.

Additionally, in engineering practice it is common to calculate overall heat transfer coefficients by adding single resistances. For this method, the resistance of the wall has to be calculated separately using simple boundary conditions as given in this paper.

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